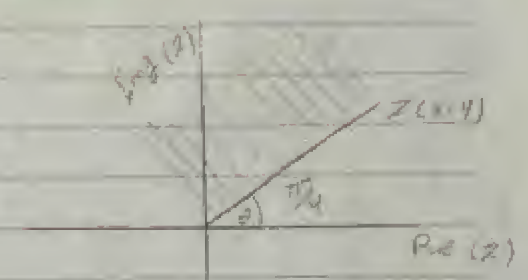


Lect 2

# Geometrical representation of equations in complex plane

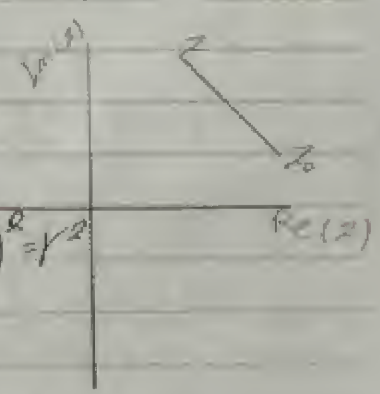
\* Arg  $Z \geq \pi/4$



$$\Rightarrow |Z - Z_0| = r$$

$$Z = (x, y) \quad Z_0 = (x_0, y_0)$$

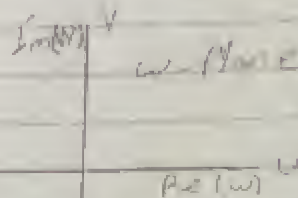
$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2} = r \Rightarrow (x - x_0)^2 + (y - y_0)^2 = r^2$$



Chapter 2

## Functions in complex analysis

 $w = f(z)$  is a complex function



$$w = f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2xyi \quad u = x^2 - y^2$$

$$w = (x^2 - y^2) + i(2xy) = u + iv \quad v = 2xy$$

express in  $u$  &  $v$  for the following

i)  $f(z) = e^z$  ~~soln~~  $w = f(z) = u + iv = e^z$  but  $z = x + iy$

$$\therefore w = e^{x+iy} = e^x \cdot e^{iy} \quad \text{but } e^{iy} = \cos y + i \sin y$$

$$w = e^x [\cos y + i \sin y] \quad \therefore w = \underbrace{e^x \cos y}_u + i \underbrace{e^x \sin y}_v$$

$$u = e^x \cos y$$

$$v = e^x \sin y$$

single-valued function  
for  $z$  in  $2\pi i$  or  $4\pi i$  or  $6\pi i$  or  $8\pi i$  or  $10\pi i$  or  $12\pi i$  or  $14\pi i$  or  $16\pi i$  or  $18\pi i$  or  $20\pi i$  or  $22\pi i$  or  $24\pi i$  or  $26\pi i$  or  $28\pi i$  or  $30\pi i$  or  $32\pi i$  or  $34\pi i$  or  $36\pi i$  or  $38\pi i$  or  $40\pi i$  or  $42\pi i$  or  $44\pi i$  or  $46\pi i$  or  $48\pi i$  or  $50\pi i$  or  $52\pi i$  or  $54\pi i$  or  $56\pi i$  or  $58\pi i$  or  $60\pi i$  or  $62\pi i$  or  $64\pi i$  or  $66\pi i$  or  $68\pi i$  or  $70\pi i$  or  $72\pi i$  or  $74\pi i$  or  $76\pi i$  or  $78\pi i$  or  $80\pi i$  or  $82\pi i$  or  $84\pi i$  or  $86\pi i$  or  $88\pi i$  or  $90\pi i$  or  $92\pi i$  or  $94\pi i$  or  $96\pi i$  or  $98\pi i$  or  $100\pi i$  or  $102\pi i$  or  $104\pi i$  or  $106\pi i$  or  $108\pi i$  or  $110\pi i$  or  $112\pi i$  or  $114\pi i$  or  $116\pi i$  or  $118\pi i$  or  $120\pi i$  or  $122\pi i$  or  $124\pi i$  or  $126\pi i$  or  $128\pi i$  or  $130\pi i$  or  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i$  or  $758\pi i$  or  $760\pi i$  or  $762\pi i$  or  $764\pi i$  or  $766\pi i$  or  $768\pi i$  or  $770\pi i$  or  $772\pi i$  or  $774\pi i$  or  $776\pi i$  or  $778\pi i$  or  $780\pi i$  or  $782\pi i$  or  $784\pi i$  or  $786\pi i$  or  $788\pi i$  or  $790\pi i$  or  $792\pi i$  or  $794\pi i$  or  $796\pi i$  or  $798\pi i$  or  $800\pi i$  or  $802\pi i$  or  $804\pi i$  or  $806\pi i$  or  $808\pi i$  or  $810\pi i$  or  $812\pi i$  or  $814\pi i$  or  $816\pi i$  or  $818\pi i$  or  $820\pi i$  or  $822\pi i$  or  $824\pi i$  or  $826\pi i$  or  $828\pi i$  or  $830\pi i$  or  $832\pi i$  or  $834\pi i$  or  $836\pi i$  or  $838\pi i$  or  $840\pi i$  or  $842\pi i$  or  $844\pi i$  or  $846\pi i$  or  $848\pi i$  or  $850\pi i$  or  $852\pi i$  or  $854\pi i$  or  $856\pi i$  or  $858\pi i$  or  $860\pi i$  or  $862\pi i$  or  $864\pi i$  or  $866\pi i$  or  $868\pi i$  or  $870\pi i$  or  $872\pi i$  or  $874\pi i$  or  $876\pi i$  or  $878\pi i$  or  $880\pi i$  or  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ii)  $w = f(z) = \ln(z)$

$$w = \ln(x+iy) = \ln(re^{i\theta}) \quad \text{if } z = re^{i\theta}$$

$$w = \ln r + \ln e^{i\theta} = \ln r + i(\theta + 2n\pi)$$

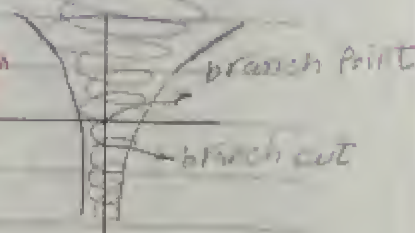
$$w = \ln r + i(\theta + 2n\pi)$$

$$u = \ln \sqrt{x^2 + y^2} \quad v = (\tan^{-1} y/x + 2n\pi)$$

at  $n=0 \Rightarrow w = \ln \sqrt{x^2 + y^2} + i \tan^{-1} y/x$  ~~included~~

$\Rightarrow$  multi-valued function

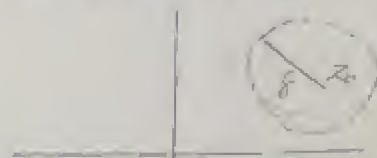
$$A = 2\pi i$$



= The limits =

$$\lim_{z \rightarrow z_0} f(z) = f(z_0) \Rightarrow |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$$

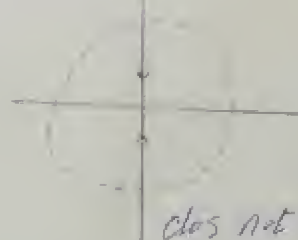
$f(z)$  has a limit at  $z_0$



ex: find  $\lim_{z \rightarrow 0} (z^2) \Rightarrow \lim_{z \rightarrow 0} [x^2 - y^2 + i(2xy)]$

\* ex:  $\lim_{z \rightarrow 0} \frac{z}{z} \Rightarrow \lim_{x \rightarrow 0} \frac{x+iy}{x+iy}$

$$\lim_{z \rightarrow 0} \frac{z}{z} = \pm 1$$



does not exist

= The continuity =

the function  $f(z)$  is continuous at  $z = z_0$

iff

1)  $f(z_0)$  exists 2)  $\lim_{z \rightarrow z_0} f(z)$  exists

$$f(z_0) = \lim_{z \rightarrow z_0} f(z)$$

ex:  $f(z) = \bar{z} \Rightarrow f(z) = x - iy$  at  $z = 0$  &  $x = 0$  &  $y = 0$   
 $f(0) = 0$

ex 2)  $\lim_{z \rightarrow 0} \bar{z} = 0$

the Derivation of the complex function

$$df/dz = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \quad \text{if the limit is existed then the function is differentiable}$$

and is defined by  $\frac{df(z_0)}{dz}$

\* every differentiable function is continuous and the

inverse isn't true

because  $\lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\Delta z}{\Delta z}$

\*  $f(z) = \bar{z}$  is continuous

but not differentiable

\* the limit isn't existed



## ANALYTIC FUNCTION

- The function  $f(z)$  is called analytic if it is differentiable at the point  $z_0$  and along its neighborhood and is called entire if it is differentiable for the whole complex plane.

Cauchy-Riemann

$$\text{I) } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{II) } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

the  $f(z) = u + iv$  is analytic and its derivative is

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

EX → Check the analyticity of the following function

1)  $f(z) = \operatorname{Re}(z^2) \rightarrow f(z) = x^2 - y^2 \quad u = x^2 - y^2 \quad v = 0$   
 $z^2 = x^2 - y^2 + 2ixy$  not analytic

2)  $f(z) = z e^{-z} \rightarrow x e^{-z} + i y e^{-z} = x e^{-x-iy} + i y e^{-x-iy}$   
 $= x e^{-x} (\cos y - i \sin y) + i y e^{-x} (\cos y - i \sin y)$

$$\begin{aligned} \text{I} &= x e^{-x} \cos y + y e^{-x} \sin y + i [x e^{-x} \sin y + y e^{-x} \cos y] \\ \therefore u &= x e^{-x} \cos y + y e^{-x} \sin y \quad v = -x e^{-x} \sin y + y e^{-x} \cos y \end{aligned}$$

$u_x = v_y$  &  $u_y = -v_x$  The function is analytic

$$\begin{aligned} \frac{df}{dz} &= u_x + i v_x \\ &= v_y - i v_x \end{aligned}$$

Cauchy-Riemann Polar Coordinates

$$\text{I) } \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \quad \text{II) } \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}$$

$$\frac{df}{dz} = (u_r + i v_r)(\cos \theta - i \sin \theta)$$